



TITLE:

# On atomic AEC and quasi-minimality (Model theoretic aspects of the notion of independence and dimension)

AUTHOR(S):

MAESONO, Hisatomo

---

CITATION:

MAESONO, Hisatomo. On atomic AEC and quasi-minimality (Model theoretic aspects of the notion of independence and dimension). 数理解析研究所講究録 2011, 1741: 46-51

ISSUE DATE:

2011-05

URL:

<http://hdl.handle.net/2433/170908>

RIGHT:

## On atomic AEC and quasi-minimality

前園 久智 (Hisatomo MAESONO)  
早稲田大学メディアネットワークセンター  
(Media Network Center, Waseda University)

### Abstract

Atomic abstract elementary class have been researched in connection with the model theory of infinitary logic. In recent years, the results were summarized by J.T.Baldin [1]. In that book, categoricity problem of atomic AEC is discussed mainly. I tried some local argument around the problem.

**Apology** In this note, I do not have exact references to the papers in which the results are originally proved.

### 1. Atomic AEC and splitting

We recall some definitions.

**Definition 1** A class of structures  $(\mathbf{K}, \prec_{\mathbf{K}})$  (of a language  $L$ ) is an *abstract elementary class* (AEC) if the class  $\mathbf{K}$  and class of pairs satisfying the binary relation  $\prec_{\mathbf{K}}$  are each closed under isomorphism and satisfy the following conditions ;

A1. If  $M \prec_{\mathbf{K}} N$ , then  $M \subseteq N$ .

A2.  $\prec_{\mathbf{K}}$  is a partial order on  $\mathbf{K}$ .

A3. If  $\{A_i : i < \delta\}$  is a  $\prec_{\mathbf{K}}$ -increasing chain :

(1)  $\bigcup_{i < \delta} A_i \in \mathbf{K}$

(2) for each  $j < \delta$ ,  $A_j \prec_{\mathbf{K}} \bigcup_{i < \delta} A_i$

(3) if each  $A_i \prec_{\mathbf{K}} M \in \mathbf{K}$ , then  $\bigcup_{i < \delta} A_i \prec_{\mathbf{K}} M$ .

A4. If  $A, B, C \in \mathbf{K}$ ,  $A \prec_{\mathbf{K}} C$ ,  $B \prec_{\mathbf{K}} C$  and  $A \subseteq B$ , then  $A \prec_{\mathbf{K}} B$ .

A5. There is a Löwenheim-Skolem number  $LS(\mathbf{K})$  such that if  $A \subseteq B \in \mathbf{K}$ , there is an  $A' \in \mathbf{K}$  with  $A \subseteq A' \prec_{\mathbf{K}} B$  and  $|A'| \leq |A| + LS(\mathbf{K})$ .

**Definition 2** We say an AEC  $(\mathbf{K}, \prec_{\mathbf{K}})$  is *atomic* if  $\mathbf{K}$  is the class of atomic models of a countable complete first order theory and  $\prec_{\mathbf{K}}$  is first order elementary submodel.

In the following,  $\mathbf{K}$  denotes an atomic AEC.

**Definition 3** Let  $T$  be a countable first order theory.

A set  $A$  contained in a model  $M$  of  $T$  is *atomic* if every finite sequence in  $A$  realizes a principal type over the empty set.

Let  $A$  be an atomic set.

$S_{at}(A)$  is the collection of  $p \in S(A)$  such that if  $a \in \mathcal{M}$  realizes  $p$ ,  $Aa$  is atomic ( where  $\mathcal{M}$  is the big model ).

We refer to a  $p \in S_{at}(A)$  as an *atomic type*.

We consider the notion of stability for atomic types.

**Definition 4** The atomic class  $\mathbf{K}$  is  $\lambda$  – *stable* if for every  $M \in \mathbf{K}$  of cardinality  $\lambda$ ,  $|S_{at}(M)| = \lambda$ .

**Example 5** ([1]) 1. Let  $\mathbf{K}_1$  be the class of atomic models of the theory of dense linear order without endpoints. Then  $\mathbf{K}_1$  is not  $\omega$ –stable.

2. Let  $\mathbf{K}_2$  be the class of atomic models of the theory of the ordered Abelian group of rationals. Then  $\mathbf{K}_2$  is  $\omega$ –stable.

The notion of independence by splitting is available in this context.

**Definition 6** A complete type  $p$  over  $B$  *splits over*  $A \subset B$  if there are  $b, c \in B$  which realize the same type over  $A$  and a formula  $\phi(x, y)$  such that  $\phi(x, b) \in p$  and  $\neg\phi(x, c) \in p$ .

Let  $A, B, C$  be atomic.

We write  $A \downarrow_C B$  and say  $A$  is *independent from*  $B$  *over*  $C$  if for any finite sequence  $a \in A$ ,  $\text{tp}(a/B)$  does not split over some finite set of  $C$ .

**Fact 7** ([1]) Under the atomic  $\omega$ –stable assumption of  $(\mathbf{K}, \prec_{\mathbf{K}})$  (and some assumption of parameters), the independence relation by splitting (over models) satisfies almost all forking axioms.

**Theorem 8** ([1]) If  $\mathbf{K}$  is  $\omega$ –stable and has a model of power  $\aleph_1$ , then it has a model of power  $\aleph_2$ .

I considered the same problems under some weaker condition.

**Definition 9** Let  $\mathbf{K}$  be an atomic AEC and  $M \in \mathbf{K}$ .

$M$  has *no infinite splitting chain* if for any  $p \in S_{at}(M)$  which is realized outside  $M$ , there is no increasing sequence  $\{A_i\}_{i < \omega} (\subset M)$  such that  $p \restriction A_{i+1}$  splits over  $A_i$  for all  $i < \omega$ .

We can prove the next facts.

**Fact 10** If  $\mathbf{K}$  is  $\omega$ –stable, then no model of  $\mathbf{K}$  has infinite splitting chain.

**Fact 11** *Let  $\mathbf{K}$  have no infinite splitting chain (i.e. every  $M \in \mathbf{K}$  has no infinite splitting chain).*

*If  $\mathbf{K}$  has a model of power  $\aleph_1$ , then it has a model of power  $\aleph_2$ .*

**Fact 12** *Under the assumption that  $(\mathbf{K}, \prec_{\mathbf{K}})$  has no infinite splitting chain, the independence relation by splitting (over models) satisfies almost all forking axioms except symmetry.*

At present, I do not have the definitive result about symmetry of splitting. But we can prove the next fact.

**Definition 13** Let  $\mathbf{K}$  be an atomic AEC and  $M \in \mathbf{K}$ .

$M$  has *infinite splitting left-chain* if there is a sequence  $\{B_i\}_{i < \omega} \subset M$  and  $b$  (outside  $M$ ), and  $A \subset M$  such that  $\text{tp}_{\text{at}}(B_i/Ab\{B_j : j < i\})$  splits over  $A \cup \{B_j : j < i\}$  for all  $i < \omega$ .

**Fact 14** *Let  $\mathbf{K}$  have no infinite splitting chain. Suppose that any countable atomic set is extended to a countable model in  $\mathbf{K}$ .*

*If the independence by splitting over models is not symmetry, then there is an infinite splitting left-chain.*

## 2. $\ast$ –excellent AEC and categoricity

We recall some definitions.

**Definition 15** The atomic AEC  $\mathbf{K}$  is  $\ast$  – *excellent* if

- A1.  $\mathbf{K}$  has arbitrarily large models,
- A2.  $\mathbf{K}$  is  $\omega$ –stable,
- A3.  $\mathbf{K}$  satisfies the amalgamation property,
- A4. Let  $p$  be a complete type over a model  $M \in \mathbf{K}$  such that  $p \restriction C$  is realized in  $M$  for each finite  $C \subset M$ , then there is a model  $N \in \mathbf{K}$  with  $N$  primary over  $M$  such that  $p$  is realized by  $a$  in  $N$ .

**Definition 16** Let  $M \in \mathbf{K}$  and  $A \subset M$ .

The type  $p \in S_{\text{at}}(A)$  is *big* if for any  $M' \supset A$  with  $M' \in \mathbf{K}$ , there exists an  $N'$  such that  $M' \prec_{\mathbf{K}} N'$  and  $p$  has a realization in  $N' - M'$ .

A triple  $(M, N, \phi)$  is called a *Vaughtian triple* if  $\phi(M) = \phi(N)$  where  $M \prec_{\mathbf{K}} N$  with  $M \neq N$  and  $L(M)$ –formula  $\phi$  is big.

The next theorem is the analogous result of Morley’s categoricity theorem for atomic AEC.

**Theorem 17** ([1]) *Suppose  $\mathbf{K}$  is an  $\ast$ –excellent atomic AEC. Then the following are equivalent.*

- (1)  $\mathbf{K}$  is categorical in some uncountable cardinality.
- (2)  $\mathbf{K}$  has no Vaughtian triple.
- (3)  $\mathbf{K}$  is categorical in every uncountable cardinality.

**Theorem 18** ([1]) For each  $2 \leq k < \omega$ , there is an  $L_{\omega_1, \omega}$ -sentence  $\phi_k$  such that :

- $\phi_k$  is categorical in  $\mu$  if  $\mu \leq \aleph_{k-2}$ , and
- $\phi_k$  is not categorical in any  $\mu$  with  $\mu > \aleph_{k-2}$ .

In the proof of Theorem 17, the geometry of quasi-minimal formula plays the important role. I considered that the argument in [5] makes the proof concise.

**Definition 19** The type  $p \in S_{at}(A)$  is *quasi-minimal* if  $p$  is big and for any  $M$  containing  $A$ ,  $p$  has a unique extension to a type over  $M$  which is not realized in  $M$ .

**Lemma 20** ([1]) Let  $\mathbf{K}$  be  $\omega$ -stable.

Then for any  $M \in \mathbf{K}$ , there is a  $c \in M$  and a formula  $\phi(x, c)$  which is quasi-minimal.

**Definition 21** Let  $X$  be an infinite set and  $\text{cl}$  a function from  $\mathcal{P}(X)$  to  $\mathcal{P}(X)$  where  $\mathcal{P}(X)$  denotes the set of all subsets of  $X$ . If the function  $\text{cl}$  satisfies the following properties, we say  $(X, \text{cl})$  is a *pregeometry*.

- (I)  $A \subset B \implies A \subset \text{cl}(A) \subset \text{cl}(B)$ ,
- (II)  $\text{cl}(\text{cl}(A)) = \text{cl}(A)$ ,
- (III) (Finite character)  $b \in \text{cl}(A) \implies b \in \text{cl}(A_0)$  for some finite  $A_0 \subset A$ ,
- (IV) (Exchange axiom)  
 $b \in \text{cl}(A \cup \{c\}) - \text{cl}(A) \implies c \in \text{cl}(A \cup \{b\})$ .

In the proof of Theorem 17, the closure operator  $\text{cl}$  is defined as follows.

**Definition 22** Let  $c \in M \in \mathbf{K}$ . And suppose  $\phi(x, c)$  determines a quasi-minimal type over  $M$ .

For any elementary extension  $N(\in \mathbf{K})$  of  $M$ ,  $\text{cl}$  is defined on the set of realizations of  $\phi(x, c)$  in  $N$  by  $a \in \text{cl}(A)$  if  $\text{tp}(a/Ac)$  is not big.

We recall some result from [5].

**Definition 23** Let  $M$  be an uncountable structure and  $p(x) \in S_1(M)$ .

For all subsets  $A \subset M$ , the operator  $\text{cl}'_p$  is defined by  $\text{cl}'_p(A) = \{a \in M : a \text{ is not a realization of } p[A]\}$ .

The  $n$ -th closure  $\text{cl}_p^n(A)$  of  $A$  is inductively defined as follows :

$$\text{cl}_p^0(A) = A \text{ and } \text{cl}_p^{n+1}(A) = \text{cl}'_p(\text{cl}_p^n(A))$$

We put  $\text{cl}_p(A) = \bigcup_{n \in \omega} \text{cl}_p^n(A)$ .

**Theorem 24** ([5]) *Let  $N$  be a quasi-minimal model and  $p(x) \in S_1(N)$ . Suppose that  $p(x)$  does not split over  $A$  for some countable  $A \subset N$  (and  $N \neq \text{cl}_p(a)$  for some finite  $a \in N$ ).*

*Then  $\text{cl}_p$  is a closure operator and exactly one of the following two holds ;*

- 1. Every  $\text{cl}_p$ -free sequence over  $A$  is totally indiscernible. In this case,  $(N, \text{cl}_p)$  is a pregeometry, and  $p$  is definable over  $A$ .*
- 2. Otherwise. In this case, there is a finite extension  $A_0$  of  $A$  and an  $A_0$ -definable partial order  $\leq$  such that every  $\text{cl}_p$ -free sequence over  $A_0$  is strictly increasing.*

We can deduce the next lemma.

**Lemma 25** *Let  $\mathbf{K}$  be  $\omega$ -stable and have a sufficiently large model. And let  $M \in \mathbf{K}$ .*

*Then there is a finite  $c \in M$  and a formula  $\phi(x, c)$  such that  $\phi(x, c)$  determines a quasi-minimal  $p(x) \in S_{\text{at}}(M)$  and  $p(x)$  does not split over  $c$ , and  $\text{cl}$  defines a pregeometry in  $(p[c](M))$ .*

### 3. $P$ -closure in atomic AEC

I considered  $P$ -closure in the quasi-minimal set  $\phi(M, c)$  above. The  $P$ -closure is the collection of realizations of types that is  $P$ -analysable and co-foreign to  $P$ . In this note, I omits the  $P$ -analysable assumption, resulting in a larger  $P$ -closure.

In this section,  $A \downarrow_C B$  means  $\text{tp}(A/BC)$  does not split over  $C$ .

#### Assumptions

Let  $\mathbf{K}$  be an  $\omega$ -stable atomic AEC and  $M \in \mathbf{K}$ .

$\phi(x, c)$  determines a quasi-minimal  $p(x) \in S_{\text{at}}(M)$  and  $p(x)$  does not split over  $c$ . And we may assume that  $c = \emptyset$ .

The set  $P$  of types is defined by

$P = \{q \in S(A') : q \text{ is a conjugate of } p \upharpoonright A \text{ for some finite } A \subset M\}$ .

**Definition 26** In this definition, parameters are finite subset of  $\phi(M, c)$  above and types are atomic types.

Now  $P$  is an  $\emptyset$ -invariant family of types.

A complete type  $q \in S_{\text{at}}(A)$  is *foreign* to  $P$  if for all  $a \models q$ ,  $A \subset B$  with  $a \downarrow_A B$ , and realizations  $\bar{c}$  of extensions of types in  $P$  over  $B$ , we always have  $a \downarrow_A \bar{c}B$ .

A partial type  $q$  is *co-foreign* to  $P$  if every type in  $P$  is foreign to  $q$ .

The  $P$ -closure  $\text{cl}_P(A)$  of a set  $A$  is the collection of all element  $a$  such that  $\text{tp}(a/A)$  is co-foreign to  $P$ .

We can prove the next fact.

**Fact 27** *Let  $\mathbf{K}$  be  $\omega$ -stable and have a sufficiently large model. And let  $M \in \mathbf{K}$  and  $\phi(x, c)$  be a quasi-minimal formula that determines  $p(x) \in S_{at}(c)$  for some  $c \in M$  as above.*

*For any  $A \subset p(M)$ ,  $cl(A) = cl_p(A)$ .*

*And  $(p(M), cl_p)$  is a pregeometry.*

## References

- [1] J.T.Baldwin, *Categoricity*, University lecture series vol. 50, AMS, 2009
- [2] S.Shelah, *Classification theory for nonelementary classes. I. the number of uncountable models of  $\psi \in L_{\omega_1, \omega}$  part A*. Israel J. of math, vol.46, pp. 212-240, 1983
- [3] S.Shelah, *Classification theory for nonelementary classes. I. the number of uncountable models of  $\psi \in L_{\omega_1, \omega}$  part B*. Israel J. of math, vol.46, pp. 241-271, 1983
- [4] O.Lessmann, *Categoricity and  $U$ -rank in excellent classes*, J. Symbolic Logic, vol.68, no.4, pp. 1317-1336, 2003
- [5] A.Pillay and P.Tanović, *Generic stability, regularity, quasi-minimality*, preprint
- [6] S.Shelah, *Classification theory*, North-Holland, 1990
- [7] F.O.Wagner, *Simple theories*, Kluwer Academic Publishers, 2000
- [8] A. Pillay, *Geometric stability theory*, Oxford Science Publications, 1996